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Grazing bifurcation and chaotic oscillations of vibro-impact systems with one degree of freedom☆

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ABSTRACT

The bifurcations of dynamical systems, described by a second-order differential equation with periodic coefficients and an impact condition, are investigated. It is shown that a continuous change in the coefficients of the system, during which the number of impacts of the periodic solution increases, leads to the occurrence of a chaotic invariant set.

The properties of dynamical systems with impact are the same in many respects as the properties of systems of ordinary differential equations. The existence, uniqueness and continuity of the solutions of vibro-impact systems have been studied[1] with respect to the initial data and parameters. Chaotic oscillations[2–12] can be observed in systems with impacts, and it has been suggested[5–10,12] that the existence of periodic solutions corresponding to a zero velocity impact is one of the causes of these. So-called grazing bifurcation[5,8–10] is associated with this phenomenon. Numerical experiments have been described which point to the possibility of the existence of a strange attractor in a half-neighbourhood of a parameter bifurcation value. It is shown below that this bifurcation leads to the formation of invariant sets which are analogous to a Smale horseshoe.[13]

1. Formulation of the problem

Consider an interval $J=[0, \mu^*]$ and a continuous function $f(t, x, y, \mu)$ which acts from $\mathbb{R}^2 \times J$ into $\mathbb{R}^2$ and is C^2-smooth with respect to its arguments. We will assume that $f(t, x, y, \mu) \equiv f(t+T(\mu), x, y, \mu)$, where $T(\mu)$ is C^2-smooth function in the interval J which is isolated from zero. The period $T(\mu)$ can be assumed to be independent of μ by making the substitution $t' = tT(0)/T(\mu)$.

Consider the system

$$\dot{x} = y, \quad \dot{y} = f(t, x, y, \mu) \tag{1.1}$$

Put

$$z = (x, y), \quad f_0(t, \mu) = f(t, 0, 0, \mu)$$

$$\Lambda = (0, +\infty) \times \mathbb{R} \cup \{0\} \times [0, +\infty) = \{(x, y) \in \mathbb{R}^2 : x > 0\} \cup \{(0, y) : y \geq 0\}$$

We will assume that system (1.1) is defined for $(t, x, y, \mu) \in \mathbb{R} \times \Lambda \times J$, and an impact, defined by the following conditions, occurs when the solutions reach the value $x=0$.

Condition C1. Suppose $z(t)=(x(t), y(t))$ is the solution of the problem. If $x(t_0)=0$, then, for a certain constant $r>0$,

$$y(t_0+0) = -ry(t_0-0) \tag{1.2}$$

Condition C2. If $z(t_0-0)=0$, $f_0(t_0, \mu) \leq 0$ and the quantity t_1 is such that $f_0(t, \mu) \leq 0$ for all $t \in I=[t_0, t_1)$, then $z(t)|_I \equiv 0$.

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We will call the resulting vibro-impact system System A and suppose $z(t, t_0, z_0, \mu)$ is the value of the solution of this system with the initial data $z(t_0) = z_0$ at an instant t (if it is defined and unique)

Conditions C1 and C2 correspond to the case of a fixed limiter. If the limiter is movable and its position is described by the function $b(t, \mu)$, the problem can be reduced to the initial problem by making the replacement

$$x_1 = x - b(t, \mu), \quad y_1 = y - \dot{b}(t, \mu).$$

System (1.1) takes the form

$$\dot{x}_1 = y_1, \quad \dot{y}_1 = f_1(t, x_1, y_1, \mu) = f(t, x_1 + b(t, \mu), y_1 + \dot{b}(t, \mu), \mu) - \ddot{b}(t, \mu),$$

and, if the function $b(t, \mu)$ is C^4-smooth and has a period T, then the function f_1, as well as f, will be C^2-smooth and will have a period T with respect to the argument t.

2. The dependence of the solutions on the initial data and the parameter

Since the velocity corresponding to the variable y changes discontinuously in the case of impacts, the integral continuity theorem is inapplicable. Nevertheless, the following two lemmas hold.

Lemma 2.1. *Suppose that, for a certain μ_0, System A has a solution $z(t) = (x(t), y(t))$ with initial data $z(t_0) = z_0$ which is defined in the interval t_-, t_+ and contains the point t_0. We will assume that the function $x(t)$ has precisely N roots $t_- < \tau_1^0 < \ldots < \tau_N^0 < t_+$ in the interval t_-, t_+ and that $y(\tau_j^0) \neq (j = 1, \ldots, N)$. A $\delta_0 > 0$ is then found such that, if $|\mu_1 - \mu_0| < \delta_0$, $|z_1 - z_0| < \delta_0$, $|t_1 - t_2| < \delta_0$, the solution $z(t, t_1, z_1, \mu_1)$ of System A, corresponding to the value μ_1 of the parameter μ and the initial data $z(t_1) = z_1$, has precisely N roots $\tau_j(t_1, z_1, \mu_1)$ $(j = 1, \ldots, N)$ in the same interval. Here, both the instants $\tau_j(t_1, z_1, \mu_1)$ and the corresponding values of the velocities $y_j = y(\tau_j(t_1, z_1, \mu_1) + 0, t_1, z_1, \mu_1)$ are C^2-smooth functions of their arguments.*

Fixing the solution $z(t)$, we introduce the notation

$$U_{\delta_0} = \{(t_1, z_1, \mu_1) : |t_1 - t_0| < \delta_0, |z_1 - z_0| < \delta_0, |\mu_1 - \mu_0| < \delta_0\}$$

Lemma 2.2. *Suppose δ_0 is a quantity which exists by virtue of Lemma 2.1 and the numbers $\tau_j^{\pm}$, defined by the formulae*

$$\tau_j^+ = \max\{\tau_j(t_1, z_1, \mu_1) : (t_1, z_1, \mu_1) \in U_{\delta_0}\}, \quad \tau_j^- = \min\{\tau_j(t_1, z_1, \mu_1) : (t_1, z_1, \mu_1) \in U_{\delta_0}\}$$

are such that

$$t_- < \tau_1^- \leq \tau_1^+ < \tau_2^- \leq \tau_2^+ < \ldots < \tau_N^- \leq \tau_N^+ < t_+$$

Then, in any of the intervals $[t_-, \tau_1^-), (\tau_1^+, \tau_2^-), \ldots, (\tau_{N-1}^+, \tau_N^-), (\tau_N^+, t_+]$, the solution $z(t, t_1, z_1, \mu_1)$ is a C^2-smooth function of its arguments, where t runs through the corresponding interval and $(t_1, z_1, \mu_1) \in U_{\delta_0}$.

We next assume that the following assertion holds.

Condition 1. A family of T-periodic solutions $\varphi(t, \mu) = (\varphi_x(t, \mu), \varphi_y(t, \mu))$ of System A exists, which depends continuously on $\mu \in J$ and possesses the following properties (Fig. 1):

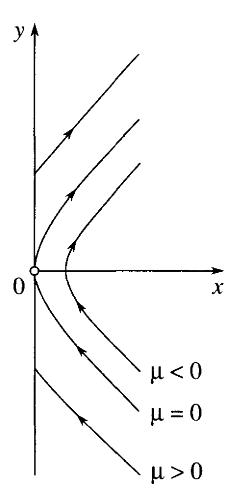

Fig. 1.

1) when $\mu > 0$, the component $\varphi_x(t, \mu)$ has exactly $N+1$ roots $\tau_0(\mu), \ldots, \tau_N(\mu)$ in a period,
2) the velocities of the impacts $y_k(\mu) = \varphi_y(\tau_k(\mu)+0, \mu)$ are such that

$$y_0(\mu) > 0 \ \text{ when } \ \mu > 0; \quad y_0(0) = 0; \quad f_0(\tau_0(0), 0) > 0$$
$$y_k(\mu) > 0 \ \forall \mu \in J, \quad k = 1, \ldots, N \tag{2.1}$$

3) the instants $\tau_k(\mu)$ and the impact velocities $y_k(\mu)$ depend continuously on the parameter μ in the domain of definition.

Without loss of generality, it can be assumed that $\tau_0(\mu) \equiv 0$. This can be achieved by making the replacement of variables $t' = t - \tau_0(\mu)$, extended in a continuous manner when $\mu < 0$. Fixing a small θ, we consider a Poincaré mapping for System A which is defined by the formula

$$F(z_0) = F_{\mu,\theta}(z_0) = z(T - \theta + 0, -\theta, z_0, \mu)$$

For sufficiently small positive μ and θ, the mapping F is C^2-smooth in a certain neighbourhood of a point $\varphi(-\theta, \mu)$. Suppose that

$$z_{\mu,\theta} = (x_{\mu,\theta}, y_{\mu,\theta}) = \varphi(-\theta + 0, \mu)$$

The mapping F generates a discrete dynamical system

$$z_{n+1} = F(z_n) \tag{2.2}$$

3. Separatrix

It follows from relations (2.1) that, in the case of small μ and θ,

$$f_0(-\theta, \mu) > 0 \tag{3.1}$$

We will denote the set of initial data of solutions which vanish at a certain instant $t_1 \in [-T/2, T/2]$ by $\Gamma_{\mu,\theta}$:

$$\Gamma_{\mu,\theta} = \{z_0 \in \Lambda : \exists t_1 \in [-T/2, T/2] : z(t_1, -\theta, z_0, \mu)\} = 0$$

Lemma 3.1. *If the parameters μ and θ are sufficiently small, the intersection of $\Gamma_{\mu,\theta}$ with a small neighbourhood of zero is a graph of the C^2-smooth function $x = \gamma_{\mu,\theta}(y)$. In addition,*

$$\gamma_{\mu,\theta}(0) = \gamma'_{\mu,\theta}(0) = 0, \quad \gamma''_{\mu,\theta}(0) = 1/f_0(-\theta, \mu) \tag{3.2}$$

Proof. We consider $\varsigma \in \Gamma_{\mu,\theta}$. Suppose the value t_0 is such that

$$z(t_0, -\theta, \varsigma) = 0; \quad s = t - t_0, \quad z(t+0, -\theta, \varsigma) = (x(t), y(t))$$

As s_0, we choose the largest number such that the function $x(t_0 + s)$ does not have roots, apart from $s = 0$, in the interval $-s_0, s_0$. Then, when $s \in (-s_0, s_0)$,

$$x(t_0 + s) = x_2 s^2 + x_3 s^3 + \ldots \tag{3.3}$$

We will show that, if the quantity t_0 is sufficiently close to $-\theta$, then $s_0 \geq |t_0 + \theta|$. If this is not so, a sequence $t_0^k \to -\theta$ (without loss of generality, we assume that $t_0^k > -\theta$), a sequence $t_1^k \in [-\theta, t_0^k]$ and a sequence of solutions $z^k(t) = (x^k(t), y^k(t))$ of System A for which $z^k(t_0^k) = 0$, $x^k(t_1^k) = 0$ are found. Regardless of whether or not the function $x^k(t)$ vanishes in the interval (t_1^k, t_0^k), instants $t_2^k \in (t_1^k, t_0^k)$ are found such that $x^k(t_2^k) = 0$ and instants $t_3^k \in (t_1^k, t_0^k)$ such that $\ddot{x}_k(t_3^k) \leq 0$. At the same time, $t_3^k \to -\theta$, $x^k(t_3^k) \to 0$, $\dot{x}^k(t_3^k) \to 0$. However, then $\ddot{x}_k(t_3^k) \to f_0(-\theta, \mu)$ which contradicts inequality (3.1).

Differentiating equality (3.3), we obtain $\dot{x}_k(t_0 + s) = 2x_2 s + 3x_3 s^2 + \ldots$. On the other hand, $x_2 = \ddot{x}(t_0)/2 \to f_0(-\theta, \mu)$ when $t_0 \to -\theta$. This means that equalities (3.2) hold.

4. Basic result

We introduce the matrix

$$A_0 = \begin{Vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{Vmatrix} = \lim_{\mu,\theta \to 0+} \frac{\partial z}{\partial z_0}(T - \theta + 0, \theta, z_0, \mu)\Big|_{z_0 = z_{\mu,\theta}}$$

into the treatment. It is correctly defined since, in the case of small positive values of μ and θ, the number of zeroes of the first component $\varphi_x(t, \mu)$ of the solution $\varphi(t, \mu)$ in the intervals $[\theta, T - \theta]$ is equal to n and the values of the corresponding velocities $\varphi_y(t, \mu)$ are non-zero. Suppose $\Delta_0 = \det A_0$. We assume that

$$\operatorname{Tr} A_0 = a_{11} + a_{22} < -\min(1, \Delta_0), \quad a_{12} > 0 \tag{4.1}$$

Theorem. *With the above assumptions, values of* $\mu_0 > 0$ and $\theta_0 > 0$ *are found such that, for any* $\mu \in (0, \mu_0)$, $\theta \in (0, \theta_0)$ *system* (2.2) *is chaotic*[14] *and, in fact, a number* $m \in \mathbb{N}$ *and a compact set* $K = K_{\mu,\theta}$*, which is invariant with respect to the mapping* F^m*, are found such that the following conditions are satisfied:*

1) *the mapping* F^m *has infinitely many periodic points in the set* $K_{\mu,\theta}$;
2) *the periodic points of the mapping* F^m *are dense everywhere in* $K_{\mu,\theta}$;
3) *a point* $p_{\mu,\theta} \in K_{\mu,\theta}$ *exists, the orbit of which* $\{F^{mn}(p_{\mu,\theta}) : n \in \mathbb{Z}\}$ *is dense in* $K_{\mu,\theta}$.

5. Grazing

We will now prove the theorem. Since all the mappings $F_{\mu,\theta}$ for a fixed value of μ are associated, it is sufficient to verify the assertion of the theorem in the case of every small μ for a certain $\theta(\mu) > 0$. In this section, we will assume that $\mu > 0$ is small and fixed. Suppose $z_0(t)$ is a solution of System A such that, t_0 we have $z(t_0 - 0) = (0, -y_0)$ at a certain instant. The quantity y_0 is assumed to be a small parameter. We put $\theta_1 = y_0^2$.

We now consider the mapping $G_1(\zeta) = z(t_0 + \theta_1 + 0, t_0 - \theta_1, \zeta)$ which is defined in the neighbourhood of the point $\zeta_0 = (\xi_0, \psi_0) = z_0(t_0 - \theta_1)$. If the value of μ is sufficiently small, then $\xi_0 > 0$, $\psi_0 < 0$. By virtue of Lemma 3.1 and the fact that the first component of the solution $z_0(t)$ vanishes at the instant t_0, we have $\xi_0 \le \psi_0^2 / f_0(-\theta_1, \mu)$.

The mapping G_1 is differentiable in the neighbourhood of the point ζ_0, and we estimate the elements of the matrix $DG_1(\zeta_0)$. We now consider a point ζ_1 which is sufficiently close to ζ_0. Suppose t_1 is the first zero of the component $x_1(t)$ of the solution after t_0 - θ

$$z_1(t) = z(t, t_0, -\theta_1, \zeta_1) = (x_1(t), y_1(t))$$

and that $y_1 = -y_1(t_1 - 0)$. If the quantity μ is small, there are no other zeroes of the function $x_1(t)$ in the interval $[t_0 - \theta_1, t_0 + \theta_1]$. We now put

$$\phi_0 = f_0(t_0, \mu)$$

$$z_+ = z_1(t_0 + \theta_1) = z(t_0 + \theta_1, t_1, 0, ry_1), \quad z_- = z_1(t_0 - \theta_1) = z(t_0 - \theta_1, t_1, 0, -y_1)$$

We agree to denote any value of η which satisfies the inequality $|\eta| \le C|y_0^m|$ by the symbol $O(y_0^m)$ (the constant C is independent of θ_1). The variational system in the case of system (1.1) has the form

$$\dot{u} = \upsilon, \quad \dot{\upsilon} = \frac{\partial f}{\partial x}(t, z_0(t), \mu)u + \frac{\partial f}{\partial y}(t, z_0(t), \mu)\upsilon \tag{5.1}$$

Since the first component $x_1(t)$ of any solution $z_1(t)$ is sufficiently close to $z_0(t)$, there is a single root $t = t_1$ in the interval $[t_0 - \theta_1, t_0 + \theta_1]$ and, according to the theorem on the differentiability of solutions with respect to the initial data, we have

$$\left.\frac{\partial z_+}{\partial(t_1, y_1)}\right|_{\substack{t_1 = t_0 \\ y_1 = y_0}} = \left\| \begin{matrix} -ry_0 + O(y_0^2) & O(y_0^2) \\ -\phi_0 + O(y_0) & r + O(y_0) \end{matrix} \right\|$$

$$\left.\frac{\partial z_-}{\partial(t_1, y_1)}\right|_{\substack{t_1 = t_0 \\ y_1 = y_0}} = \left\| \begin{matrix} y_0 + O(y_0^2) & O(y_0^2) \\ -\phi_0 + O(y_0) & -1 + O(y_0) \end{matrix} \right\|$$

Then,[8]

$$B_1 = \left.\frac{\partial z_+}{\partial z_-}\right|_{\substack{t_1 = t_0 \\ y_1 = y_0}} = \left\| \begin{matrix} -r + O(y_0) & O(y_0) \\ -(r+1)\phi_0(1 + O(y_0))/y_0 & -r + O(y_0) \end{matrix} \right\| \tag{5.2}$$

Here, $\det B_1 = r^2 + O(y_0)$. Suppose $t_- < t_0 < t_+$ and R is a small square, lying in the small left half-neighbourhood of the curve Γ. It then follows from equality (5.2) that the set $R_1 = \{z(t_+, t_-, z_0, \mu) : z_0 \in R\}$ has the form shown in Fig. 2.

6. Estimates of the Lyapunov exponents

For a fixed value of $\mu > 0$, we put

$$\theta_1 = y_0^2(\mu), \quad D_1 = DF(z_{\mu,\theta_1})$$

We represent the mapping F in the form of the composition

$$F(\zeta) = F_2(F_1(\zeta)); \quad F_1(\zeta) = z(\theta_1 + 0, -\theta_1, \zeta, \mu), \quad F_2(\zeta) = z(T - \theta_1 + 0, \theta_1, \zeta, \mu).$$

The matrix $DF_2(F_1(z_{\mu,\theta_1}))$ tends to A_0 when $\mu \to 0$, and the matrix $DF_1(z_{\mu,\theta_1})$ has the form of (5.2), where $y_0 = y_0(\mu)$, $\phi_0 = f_0(0, \mu)$.

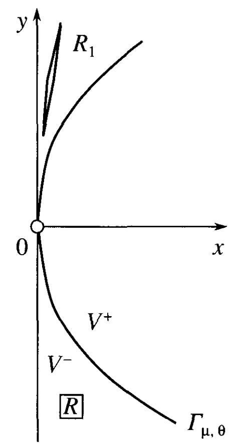

Fig. 2.

We introduce the notation

$$b_{j1} = -(r+1)a_{j2}\phi_0/y_0, \quad b_{j2} = -ra_{j2}; \quad j = 1, 2$$

Then,

$$D_1 = \left\| \begin{matrix} b_{11}(1+O(y_0)) & b_{12}(1+O(y_0)) \\ b_{21}(1+O(y_0)) & b_{22}(1+O(y_0)) \end{matrix} \right\| \tag{6.1}$$

Since $\det D_1 = -r\Delta_0 + O(y_0)$, when inequalities (4.1) are satisfied and in the case of small μ, the matrix D_1 has the eigenvalues

$$\lambda_1 = b_{11}(1+O(y_0)), \quad \lambda_2 = r^2\Delta_0(1+O(y_0))/b_{11} \tag{6.2}$$

We will now agree, when speaking of vectors, to have columns in mind but to write them in the form of rows. The eigenvector u_1, corresponding to the eigenvalue λ_1, is equal to $(a_{12}+O(y_0), a_{22}+O(y_0))$. Since $a_{12} \neq 0$, this vector is not collinear with the vector (0,1). The matrix D_1^{-1} has the form

$$\frac{1}{r^2\Delta_0}\left\| \begin{matrix} b_{22}(1+O(y_0)) & -b_{21}(1+O(y_0)) \\ -b_{12}(1+O(y_0)) & b_{11}(1+O(y_0)) \end{matrix} \right\| \tag{6.3}$$

whence it follows that the eigenvector of the matrix D_1^{-1}, corresponding to λ_2^{-1}, satisfies the asymptotic estimate $u_2 = (O(y_0), 1 + O(y_0))$. It will also be the eigenvector of the matrix D_1 corresponding to λ_2.

We now consider $\theta_2 = y_0(\mu)^{1/2}$. Suppose $\Phi(t)$ is the fundamental matrix of system (5.1), which satisfies the condition $\Phi(\theta_1) = E$. Then,

$$\Phi(\theta_1 \pm \theta_2) = E + O(\theta_2)$$

It is obvious that the eigenvalues $\hat{\lambda}_1$ and $\hat{\lambda}_2$ of the matrix $D_2 = DF(z_{\mu,\theta_2})$ satisfy asymptotics which differ from (6.2) in the replacement of $O(y_0)$ by $O(\theta_2)$ while the limiting directions of the corresponding eigenvectors are the same as in the case of the matrix D_1. We next consider $\theta = \theta_2$. We note that, for any point p from a sufficiently small neighbourhood $z_{\mu,\theta}$ in the case of small μ, the matrices $DF(p)$ and $DF^{-1}(p)$ have the form (6.1) and (6.3) respectively, with replaced $O(y_0)$ by $O(\theta_2)$ and their eigenvalues satisfy the asymptotic estimates (6.2) with the same changes. An analogous assertion also holds for the eigenvectors.

7. Homoclinic point

Consider the sets

$$V = \{(x,y) \in \Lambda : |x - x_{\mu,\theta}| \le x_{\mu,\theta}/2, \; |y - y_{\mu,\theta}| \le |y_{\mu,\theta}|/2\}$$

$$V^- = \{(x,y) \in V : x < \gamma_{\mu,\theta}(y)\}, \quad V^+ = \{(x,y) \in V : x > \gamma_{\mu,\theta}(y)\}$$

The mapping F is differentiable at the points of the set V^- and the moduli of the eigenvalues of the corresponding Jacobian matrix are not equal to unity if the parameter μ is small. Then, by virtue of Perron's theorem, a stable manifold W^s and an unstable manifold W^u of the mapping F exist in the neighbourhood of the point $z_{\mu,\theta}$. They are both curves which are smooth in the neighbourhood of the point W^s and the corresponding tangential vectors at the point $z_{\mu,\theta}$ are equal to (0, 1) and (a_{12}, a_{22}), apart from quantities of the order of $O(\theta)$. We extend these manifolds up to the invariant manifolds and, here, the resulting sets, generally speaking, will consists of not more than a denumerable number of piecewise-smooth curves.

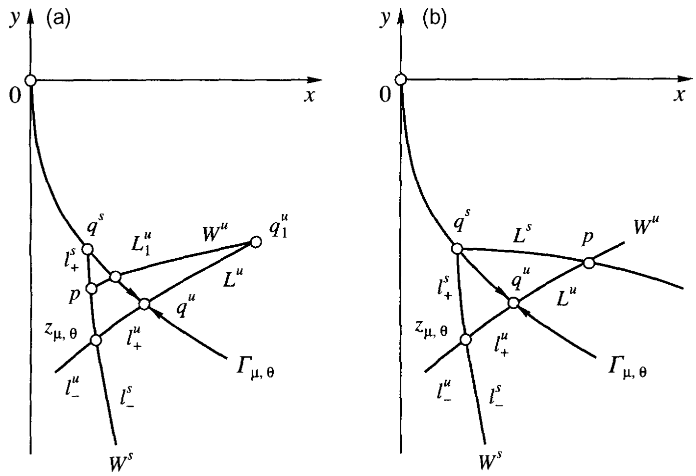

Fig. 3.

In the neighbourhood of $z_{\mu,\theta}$, we introduce the coordinates $\tilde{x}$ and $\tilde{y}$, which depend smoothly on x and y, such that the following conditions are satisfied:

1) the point $z_{\mu,\theta}$ corresponds to the values $\tilde{x} = \tilde{y} = 0$;
2) $\lim\limits_{\mu,\theta\to 0+} \dfrac{\partial(\tilde{x},\tilde{y})}{\partial(x,y)}(z_{\mu,\theta}) = \left\| \begin{matrix} 1 & 0 \\ a_{12} & -a_{22} \end{matrix} \right\|$,
3) in a certain neighbourhood of the point $z_{\mu,\theta}$, the curves defined by the conditions $\tilde{y} = 0$ and $\tilde{x} = 0$ are subsets of the manifolds W^s and W^u respectively.

Lemma 7.1. *For small values of* μ, *the manifolds* W^s *and* W^u *transversely intersect at a certain point* $p \neq z_{\mu,\theta}$ (Fig. 3).

Proof. We put

$$\sigma = \operatorname{sign} a_{12}, \quad \kappa = \sqrt{a_{12}^2 + a_{22}^2}$$

The manifold W^u intersects the curve $\Gamma_{\mu,\theta}$ at a certain point q^u. We denote the arcs of the manifold W^u, bounded by the point $z_{\mu,\theta}$ on one side and by the curve $\Gamma_{\mu,\theta}$ (or the Oy axis) on the other side, by the symbols l_+^u and l_-^u (Fig. 3, a). We denote one of the arcs $F(l_\pm^u)$ which contains the point q^u, by L^u and its length by δ^u and define $q_1^u = F(q^u)$. Suppose l_+^s and l_-^s are arcs of the manifold W^s, which are bounded on one of the sides by the point $z_{\mu,\theta}$ and lie as a whole in the domain V^-. We denote the smaller of their lengths by the symbol δ^s.

Suppose d_{ij} are the elements of the matrix A_0^2. In the case of small μ, the point q_1^u may be as close to $z_{\mu,\theta}$ as desired but, at the same time, lies in the domain V^+. Then, when $\mu \to 0$, the matrix $DF(q_1^u)$ can be as close as desired to the matrix A_0 and the product of this matrix and the vector $\nu = (a_{12}, a_{22})$ can be as close as desired to the vector (d_{12}, d_{22}). In the neighbourhood of the point q_1^u, the manifold W^u is not smooth. When $\mu \to 0$, the vectors which are tangential to the curve $L_1^u = F(L^u)\backslash L^u$ uniformly tend to the vector (d_{12}, d_{22}). The existence in the case of the system

$$\sigma\delta^u(a_{11}, a_{22}) + \sigma\delta^u(d_{12}, d_{22})X = \kappa\delta^s(0, 1)Y$$

of a solution $(X_0, Y_0) \in (0, 1) \times (-\infty, 1)$ is a sufficient condition for the curve L_1^u to intersect the manifold W^s for small μ. It can be verified by direct calculations that

$$X_0 = -1/\mathrm{Tr}A_0, \quad Y_0 = \sigma\delta^u\Delta_0/(\delta^s\kappa\mathrm{Tr}A_0)$$

A solution of the required form is then found if $\mathrm{Tr}A_0 < -1$ and $a_{12} > 0$. We shall call these conditions Case A.

Another mechanism for the appearance of a homoclinic point is possible (Fig. 3, b). Suppose the manifold W^s intersects the curve $\Gamma_{\mu,\theta}$, and q^s is the point of the corresponding intersection. The set $F^{-1}(l_-^s)$ then contains the arc L^s, one of the ends of which is the point q^s, and the tangential vectors at any of its points uniformly tend to the vector $A_0^{-1}(0, 1) = 1/\Delta_0(-a_{12}, a_{11})$. The existence in the case of the system

$$\kappa\delta^s(0, 1) + \delta^s(-a_{12}, a_{11})X = \sigma\delta^m(a_{12}, a_{22})Y$$

of a solution $(X_1, Y_1) \in (0, 1) \times (-\infty, 1)$ is a sufficient condition for the curves L^s and L^u to intersect for small μ. It can be verified by direct calculations that

$$X_1 = -\Delta_0/\mathrm{Tr}A_0, \quad Y_1 = \sigma\delta^s\kappa/(\delta^u\mathrm{Tr}A_0)$$

Then, L^s and L^u intersect if $\mathrm{Tr}A_0 < -\Delta_0$ and $a_{12} > 0$. We shall call these conditions System B. By virtue of condition (4.1), one of the cases A or B holds if $a_{12} > 0$. The lemma is proved.

8. Symbolic dynamics

The Smale–Birkhoff theorem[14] on the existence of a chaotic set in the neighbourhood of a homoclinic point is formally inapplicable in this case since the mapping F is discontinuous. Nevertheless, an assertion which is analogous to the above mentioned theorem also holds in the case being considered. We shall assume below, without loss of generality, that Case B holds and, at the same time, $a_{12} > 0$. Case A is treated in a similar manner.

Consider the neighbourhood Q_0 of the point $z_{\mu,\theta}$, defined by the conditions $|\tilde{x}| \leq \varepsilon_s$, $|\tilde{y}| \leq \varepsilon_u$ (Fig. 4) and denote the boundaries of Q_0 corresponding to $\tilde{x} = \pm\varepsilon_s$ and $\tilde{y} = \pm\varepsilon_u$ by $\partial_x^\pm$ and $\partial_y^\pm$. Assuming that n is an integer, we put $Q_n = F_n(Q_0)$. We select positive values of ε_s and ε_u such that the inequality $\varepsilon_s \geq 2\varepsilon_u$ is satisfied and, at the same time, positive numbers m^+ and m^- are found such that

$$Q_{-m^-} \cap V^- \supset l_+^s \cup l_-^s, \quad Q_{m^+} \cap V^- \supset l_+^u \cup l_-^u, \quad p \in Q_{m^+}, \quad Q_j = V^-$$

for any $-m^- < j < m^+$.

By virtue of Lemma 7.1, the set $Q_{-m^-} \cap Q_{m^+}$ comprises the minimum of the two connected components, one contains the point $z_{\mu,\theta}$ and the other contains the point p. We denote these components by $\tilde{H}_0$ and $\tilde{H}_1$ and put $m = m^+ + m^-$, $F^m(\tilde{H}_0)$, $H_1 = F^m(\tilde{H}_1)$, $H = H_0 \cup H_1$ (Fig. 5). We will show that the set $K = \bigcap_{n \in \mathbb{Z}} F^{mn}(H)$ satisfies all the requirements of the theorem which has been proved. It is obvious that the set K is invariant under the mapping F^m and that it is compact and non-empty since it contains the point $z_{\mu,\theta}$. Moreover, it does not intersect either the pre-images of the Oy axis under the mappings $F^{mk} (k \in \mathbb{Z})$ or the pre-images of the curve $\Gamma_{\mu,\theta}$ under the same mappings. Consequently, in the case of integer n, a neighbourhood Ω of the set K exists such that $F^{mn}|_\Omega$ are C^2-smooth mappings.

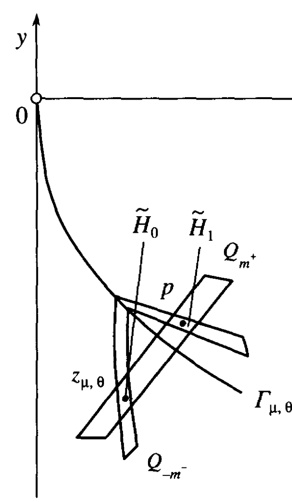

Fig. 4.

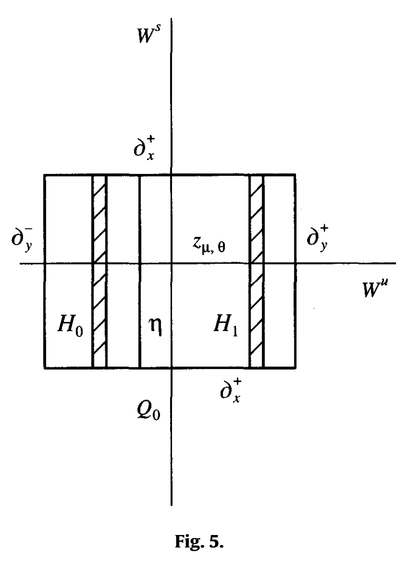

Fig. 5.

Lemma 8.1. *For any $N \in \mathbb{N}$ and any set of numbers $i = (i_0, \ldots, i_N)$ such that $i_k \in \{0, 1\}$, the set*

$$H_i = H_{i_0} \cap F^m(H_{i_1}) \cap \ldots \cap F^{mN}(H_{i_m})$$

is non-empty for any $k = 0, \ldots, N$.

Proof. Consider an arbitrary curve η joining the segments ∂_x^+ and ∂_x^- of the boundary of the set Q_0, which is the graph of the C^1-smooth function $\tilde{x} = h(\tilde{y})$ such that

$$\max|h'(\tilde{y})| \le 1 \tag{8.1}$$

Repeating the proof of the λ-lemma, well-known in the theory of structurally stable systems (Ref. 15, Theorem 6.1), it can be shown that, if μ, ε_s and ε_u are sufficiently small, an inclusion of the curve $F^m(\eta)$ in the set W^s exists which can be as close as desired to an identity inclusion in the metric C^1. In particular, this means that the curve $F^{-m}(\eta)$ contains two arcs: η_0 and η_1 which join the segments ∂_x^+ and ∂_x^- and satisfy condition (8.1). We now fix the index i. It follows from what has been said above that the pre-image of any curve η, which satisfies condition (8.1), contains the curve $\eta_{i_0} \subset H_{i_0}$ joining the segments ∂_x^+ and ∂_x^-. Applying the same reasoning to η_{i_0}, we obtain that the curve $\eta_{i_0 i1} \subset F^{-m}(\eta_{i_0}) \cap H_{i_1}$ exists. As the final result, we obtain the curve

$$\eta_{i_0 i_1 \ldots i_N} \subset F^{-mN}(H_{i_0}) \cap F^{-mN+m}(H_{i_1}) \cap \ldots \cap H_{i_N}$$

The assertion of the lemma then follows from the fact that $F^{mN}(\eta_{i_0 i_1 \ldots i_N}) \subset H_i$. A unique sequence

$$i(z) = \{\ldots, i_{-2}, i_{-1}, i_0, i_1, i_2, \ldots\}$$

such that $F^{mn}(z) \in H_{i_n}$ for any $n \in \mathbb{Z}$, corresponds to each point $z \in K$. It follows from Lemma 8.1 that the corresponding point z can be picked out for any sequence i. By virtue of the hyperbolic form of the diffeomorphism F^m in the neighbourhood of the set K, the point $z \in K$ is uniquely defined by the sequence $i(z)$. Thus, the mapping F^m in the set K is topologically associated with a mapping of a left-shift in a set of sequences of zeroes and ones, which are infinite on both sides, which shows that the assertion of the theorem holds.

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